Article

Series of Floor and Ceiling Function—Part II: Infinite Series

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Abstract: In this part of the series of two papers, we extend the theorems discussed in part I for infinite series. We then use these theorems to develop distinct novel results involving the Hurwitz zeta function, Riemann zeta function, Polylogarithm and Fibonacci numbers. In continuation, we obtain some zeros of the newly developed zeta functions and explain their behaviour using plots in complex plane. Furthermore, we provide particular cases for the theorems and corollaries which show that our results generalise the currently available functions and series such as the Riemann zeta function and the geometric series. Finally, we provide four miscellaneous examples to showcase the vast scope of the developed theorems.

Keywords: ceiling function; floor function; Fibonacci Number; Generalised Dirichlet series; Lerch−Zeta Function; Hurwitz − Zeta function; Polylogarithm; Riemann−Zeta function

MSC: 11M06; 11M26; 11M35; 30E20; 33E20

1. Introduction

In the year 1650, Pietro Mengoli posed a mathematical problem which is now known as "Basel Problem". Its solution was achieved nearly 85 years later in 1735, by Leonard Euler [1] who used an ingenious way of utilizing Taylor's series of the sine function and then generalises the formula for all real powers greater than 1. The approach, however, to implement fundamental theorem of algebra (which is for finite zeros) on an infinite polynomial (with infinite zeros) was based on Euler's intuition and remained unproved for another century when in early 1800s, Wierstrass gave validation of Euler's work using the so called "Weierstrass factorization theorem". Following these findings, in 1859, nearly two centuries after Mengoli's work, Bernard Riemann extended the formula defined by Euler to the domain of complex numbers with a motivation to find a relation between zeros of the function and the distribution of prime numbers.

The slow progress in the development of this field over the first two centuries suddenly surged after Riemann's study displayed the (zeta) function's relation with the prime counting function. Over the next two centuries researchers discovered that not only does the zeta function is crucial in understanding the distribution of prime numbers, but the function and its generalisations have many direct or indirect applications in many advanced fields such as cryptography (applications of prime number theory), cosmology [2], quantum field theory [3,4] and string theory [5].

Due to such impactful applications, researchers have studied different infinite series and zeta functions in depth [6–12] over past few decades. Coffey [6] obtained faster convergent series representation of the Hurwitz Zeta function, whereas Kanemitus et al. [7] provided integral representations and gave proofs of certain available results including Ramanujan formula. Vepštas [8] provided a technique to obtain faster convergence of oscillatory sequences and applied them to Hurwitz zeta function and Polylogarithm.

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Laurinčikas and Šiaučiūnas [9] have considered a Dirichlet series and shown its relation with Riemann zeta function. Kalimeris and Fokas [10] derived a new integral equation for the second power of the absolute value of Riemann zeta function. Nisar [11] generalised the Hurwitz-Lerch Zeta function of two variables. Riguidel [12] utilised the computational approach to propose the morphogenesis interpretation of Riemann zeta function.

Continuing the studies in the similar direction, in this part, we have attempted to generalise the different infinite series and zeta functions (such as geometric series, Hurwitz Zeta function, Polylogarithm) using the theorems developed in Part I [13].

Outline of the Article

Section 2 contains the preliminary results that are utilised in our study. Section 3 provides the cases for $n \to \infty$ for "Floor-Ceiling theorem" and "Ceiling-Floor theorem" of part I [13]. These cases provide foundations for the results obtained in sections 4-6. Section 7 is specifically the corollaries of the results proved in previous three sections. In section 8, some of the zeros of newly developed zeta functions are given and their behaviour are discussed in a bounded interval using domain colouring. Section 9 gives results on particular values. Section 10 provides some miscellaneous results. Section 11 discusses the scope for further studies and finally section 12 concludes the work done in pair of both the articles.

2. Preliminaries

The following results and definitions are useful for our study:

2.1. Hurwitz−Zeta function

The Hurwitz−Zeta function [3] $\zeta(s,t)$ is a function of a complex variables s and t defined as infinite sum:

$$\zeta(s,t) = \sum_{n=0}^{\infty} \frac{1}{(n+t)^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}e^{-tx}}{1-e^x} dx, where\ \Gamma(s) = \int_0^{\infty} x^{s-1}e^{-x}dx. \tag{1}$$

The series is absolutely convergent for all complex value of s and t when $Re(s) > 1$ and $y \in \mathbb{C}\backslash\mathbb{Z}^-$.

2.2. Polylogarithm

Polylogarithm function [14] is an infinite series of the form:

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}, \tag{2}$$

where, $|z| < 1,\ y \in \mathbb{C}\backslash\mathbb{Z}_0^-, \Re(s) > 0$.

2.3. Riemann−Zeta function

The Riemann−Zeta function [5] $\zeta(s)$ is a function of a complex variable s defined as infinite sum:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx,\ where\ \Gamma(s) = \int_0^{\infty} t^{s-1}e^{-t}dt. \tag{3}$$

The function converges for all complex value of s when $Re(s) > 1$ and defines as

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

2.4. Fibonacci Numbers and Reciprocal Fibonacci Constant

The n^{th} Fibonacci number [15] is given by the formula:

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$$F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}} \textit{ where } \varphi = \frac{1+\sqrt{5}}{2}$$

Furthermore, series of reciprocals of all Fibonacci numbers gives a constant (irrational) value:

$$\sum_{n=1}^{\infty} \frac{1}{F_n} = 3.35988566243\ldots, \tag{4}$$

where $3.35988566243\ldots$ is the reciprocal Fibonacci-Constant.

2.5. Floor and Ceiling functions

The floor function [16] of any real number x (denoted by $\lfloor x \rfloor$) gives the greatest integer not greater than x, i.e., $\lfloor x \rfloor = max\{w \in \mathbb{Z} \mid w \leq x\}$. For example, $\lfloor 1.4 \rfloor = 1$, $\lfloor 2 \rfloor = 2$, $\lfloor -3.4 \rfloor = -4$ and $\lfloor -2 \rfloor = -2$.
The ceiling function [16] (denoted by $\lceil x \rceil$), same way gives the smallest integer not smaller than x, i.e., $\lceil x \rceil = \min\{k \in \mathbb{Z} \mid k \geq x\}$. For example, $\lceil 1.4 \rceil = 2$, $\lceil 2 \rceil = 2$, $\lceil -3.4 \rceil = -3$ and $\lceil -2 \rceil = -2$.
From above, we can see that $\lceil x \rceil = \lfloor x \rfloor = x$ if and only if $x \in \mathbb{Z}$.

3. Foundations

3.1. Floor-Ceiling Theorem

Theorem 1. *Let $a, b \in \mathbb{R}^+$, $m \in \mathbb{Z}$ and let k_n be any sequence then*

$$\sum_{n=1}^{\infty} k_{\lfloor (bn)^a \rfloor + m} = \sum_{n=1}^{\infty} \left[\left\lceil \frac{(n+1)^{\frac{1}{a}}}{b} \right\rceil - \left\lceil \frac{n^{\frac{1}{a}}}{b} \right\rceil \right] k_{n+m} \tag{5}$$

or alternatively

$$\sum_{n=1}^{\infty} k_{\lfloor (bn)^a \rfloor + m} = \sum_{n=1}^{\infty} \left[(k_{n-1+m} - k_{n+m}) \left\lceil \frac{n^{\frac{1}{a}}}{b} \right\rceil \right] - \left\lceil \frac{1}{b} \right\rceil k_m \tag{6}$$

Proof. We have ([13], section 3 equation 12)

$$\sum_{i=1}^{n} k_{(\lfloor (bi)^a \rfloor + m)} = \sum_{t=1}^{\lfloor (bn)^a \rfloor} \left[\left\lceil \frac{(t+1)^{\frac{1}{a}}}{b} \right\rceil - \left\lceil \frac{t^{\frac{1}{a}}}{b} \right\rceil \right] k_{t+m} - \left(\left\lceil \frac{(\lfloor (bn)^a \rfloor)^{\frac{1}{a}}}{b} \right\rceil - n \right) k_{\lfloor (bn)^a \rfloor + m}$$

Now the following observation can be easily made:

$$\left\lceil \frac{(\lfloor (bn)^a \rfloor)^{\frac{1}{a}}}{b} \right\rceil \sim n \quad as \quad n \to \infty \ \Rightarrow \lim_{n \to \infty} \left(\left\lceil \frac{(\lfloor (bn)^a \rfloor)^{\frac{1}{a}}}{b} \right\rceil - n \right) = 0$$

Hence by applying limit $n \to \infty$ the previous equation reduces to equation (5).
Furthermore, consider the right hand side of equation (5):

$$\sum_{n=1}^{\infty} \left[\left\lceil \frac{(n+1)^{\frac{1}{a}}}{b} \right\rceil - \left\lceil \frac{n^{\frac{1}{a}}}{b} \right\rceil \right] k_{n+m} = \sum_{n=1}^{\infty} \left\lceil \frac{(n+1)^{\frac{1}{a}}}{b} \right\rceil k_{n+m} - \sum_{n=1}^{\infty} \left\lceil \frac{n^{\frac{1}{a}}}{b} \right\rceil k_{n+m}$$
$$= \sum_{n=2}^{\infty} \left\lceil \frac{n^{\frac{1}{a}}}{b} \right\rceil k_{n-1+m} - \sum_{n=1}^{\infty} \left\lceil \frac{n^{\frac{1}{a}}}{b} \right\rceil k_{n+m} = \sum_{n=1}^{\infty} \left\lceil \frac{n^{\frac{1}{a}}}{b} \right\rceil k_{n-1+m} - \left\lceil \frac{1}{b} \right\rceil k_m - \sum_{n=1}^{\infty} \left\lceil \frac{n^{\frac{1}{a}}}{b} \right\rceil k_{n+m}$$

which is the right hand side of the equation (6). □

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3.2. Ceiling-Floor Theorem

Theorem 2. *Let $a, b \in \mathbb{R}^+$, $m \in \mathbb{Z}$ and let k_n be any sequence then:*

$$\sum_{n=1}^{\infty} k_{\lceil (bn)^a \rceil + m} = \sum_{n=1}^{\infty} \left[\left\lfloor \frac{n^{\frac{1}{a}}}{b} \right\rfloor - \left\lfloor \frac{(n-1)^{\frac{1}{a}}}{b} \right\rfloor \right] k_{n+m} \tag{7}$$

or alternatively,

$$\sum_{n=1}^{\infty} k_{\lceil (bn)^a \rceil + m} = \sum_{n=1}^{\infty} \left[(k_{n+m} - k_{n+1+m}) \left\lfloor \frac{n^{\frac{1}{a}}}{b} \right\rfloor \right] \tag{8}$$

Proof. We have ([13], section 3, equation 16)

$$\sum_{i=1}^{n} k_{\lceil (bi)^a \rceil + m} = \sum_{t=1}^{\lceil (bn)^a \rceil} \left[\left\lfloor \frac{t^{\frac{1}{a}}}{b} \right\rfloor - \left\lfloor \frac{(t-1)^{\frac{1}{a}}}{b} \right\rfloor \right] k_{t+m} - \left(\left\lfloor \frac{(\lceil (bn)^a \rceil)^{\frac{1}{a}}}{b} \right\rfloor - n \right) k_{\lceil (bn)^a \rceil + m}$$

Now the following observation can be easily made:

$$\left\lfloor \frac{(\lceil (bn)^a \rceil)^{\frac{1}{a}}}{b} \right\rfloor \sim n \quad as \quad n \to \infty \Rightarrow \lim_{n \to \infty} \left(\left\lfloor \frac{(\lceil (bn)^a \rceil)^{\frac{1}{a}}}{b} \right\rfloor - n \right) = 0$$

Hence by applying limit $n \to \infty$ the previous equation reduces to equation (7).
Furthermore, consider the right hand side of equation (7):

$$\sum_{n=1}^{\infty} \left[\left\lfloor \frac{n^{\frac{1}{a}}}{b} \right\rfloor - \left\lfloor \frac{(n-1)^{\frac{1}{a}}}{b} \right\rfloor \right] k_{n+m} = \sum_{n=1}^{\infty} \left\lfloor \frac{n^{\frac{1}{a}}}{b} \right\rfloor k_{n+m} - \sum_{n=1}^{\infty} \left\lfloor \frac{(n-1)^{\frac{1}{a}}}{b} \right\rfloor k_{n+m}$$
$$= \sum_{n=1}^{\infty} \left\lfloor \frac{n^{\frac{1}{a}}}{b} \right\rfloor k_{n+m} - \sum_{n=0}^{\infty} \left\lfloor \frac{n^{\frac{1}{a}}}{b} \right\rfloor k_{n+1+m} = \sum_{n=1}^{\infty} \left\lfloor \frac{n^{\frac{1}{a}}}{b} \right\rfloor k_{n+m} - \sum_{n=1}^{\infty} \left\lfloor \frac{n^{\frac{1}{a}}}{b} \right\rfloor k_{n+1+m}$$

which is the right hand side of the equation (8). □

4. 'F' and 'C' Generalised Functions

4.1. Definitions

In order to avoid misunderstanding of our new results with the available definitions we assign $'F'$ or $'C'$ (representing use of Floor and Ceiling functions respectively) prefix to all of the available definition.
For example:
F-Hurwitz zeta function can be defined as the following equation:

$${}^{F}\zeta_b^a(s,t) = \sum_{n=0}^{\infty} \frac{1}{(\lfloor (bn)^a \rfloor + t)^s}$$

whereas, C-Riemann Zeta Function is defined as the following equation:

$${}^{C}\zeta_b^a(s) = \sum_{n=1}^{\infty} \frac{1}{\lceil (bn)^a \rceil^s}$$

Using the same analogy, "C-Hurwitz Zeta Function", "F-Polylogarithm", "C-Polylogarithm" and "F-Riemann Zeta Function" can also be defined.
Furthermore, by implementing Floor and Ceiling functions any available infinite series can be generalised as $'F'$ or $'C'$ series (i.e., For Lerch zeta function the corresponding "F-Lerch" and "C-Lerch" zeta functions can be defined).

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4.2. Theorems

For each $'F'$ and $'C'$ generalised series, a corresponding theorem can be provided and proved which relate them to an equivalent series (which is due to theorem 1 or theorem 2). Take "F-Hurwitz Zeta function" and "C-Polylogarithm" for example:

Theorem 3. *(F-Hurwitz Zeta Function)*
The series has an equivalent form given as below:

$$^{F}\zeta_b^a(s,t) = \sum_{n=0}^{\infty} \frac{1}{(\lfloor (bn)^a \rfloor + t)^s} = \sum_{n=1}^{\infty} \frac{\left[\left\lceil \frac{(n+1)^{\frac{1}{a}}}{b} \right\rceil - \left\lceil \frac{n^{\frac{1}{a}}}{b} \right\rceil\right]}{(n+t)^s} \tag{9}$$

Proof. Equation (9) can be obtained by substituting $m = 0$ and $k_n = \frac{1}{(n+t)^s}$ in equation (5). □

Theorem 4. *(C-Polylogarithm)*
The series has an equivalent form given as below:

$$^{C}Li_{s_b}^a(z) = \sum_{n=1}^{\infty} \frac{z^{\lceil (bn)^a \rceil}}{\lceil (bn)^a \rceil^s} = \sum_{n=1}^{\infty} \frac{\left[\left\lfloor \frac{n^{\frac{1}{a}}}{b} \right\rfloor - \left\lfloor \frac{(n-1)^{\frac{1}{a}}}{b} \right\rfloor\right] z^n}{n^s} \tag{10}$$

Proof. Equation (10) can be obtained by substituting $m = 0$ and $k_n = \frac{z^n}{n^s}$ in equation (7). □

Remark 1. *Every series or function available in the literature can be generalised using the same analogy. However, to avoid the repetition we have provided just a couple of examples to depict the scope of theorems 1 and 2 (General rule for the notation is $^{F}G_b^a(X)$ and $^{C}G_b^a(X)$ for $'F'$ and $'C'$ generalised functions respectively, where $G(X)$ is the notation for the regular definition of the same function).*

5. Generalised Geometric Series

5.1. Floor-Geometric Series

Theorem 5. *Let $x \in \mathbb{C}, a, b \in \mathbb{R}^+, n \in \mathbb{N}$ and $|z^x| < 1$, then the following equation holds true.*

$$\sum_{n=1}^{\infty} z^{\lfloor (bn)^a \rfloor x} = (z^{-x} - 1) \sum_{n=1}^{\infty} z^{nx} \left\lceil \frac{n^{\frac{1}{a}}}{b} \right\rceil - \left\lceil \frac{1}{b} \right\rceil \tag{11}$$

Proof. Substituting $m = 0$ *and* $k_n = z^{n \cdot x}$ in equation (6), we get

$$\sum_{n=1}^{\infty} z^{\lfloor (bn)^a \rfloor x} = \sum_{n=1}^{\infty} \left[z^{(n-1)x} - z^{nx}\right] \left\lceil \frac{n^{\frac{1}{a}}}{b} \right\rceil - \left\lceil \frac{1}{b} \right\rceil$$

Further, with a basic manipulations we arrive at equation (11). □

5.2. Ceiling-Geometric Series

Theorem 6. *Let $x \in \mathbb{C}, a, b \in \mathbb{R}^+, n \in \mathbb{N}$ and $|z^x| < 1$ then following equation holds true.*

$$\sum_{n=1}^{\infty} z^{\lceil (bn)^a \rceil x} = (1 - z^x) \sum_{n=1}^{\infty} z^{nx} \left\lfloor \frac{n^{\frac{1}{a}}}{b} \right\rfloor \tag{12}$$

Proof. Substituting $m = 0$ *and* $k_n = z^{n \cdot x}$ in equation (8), we get

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$$\sum_{n=1}^{\infty} z^{\lceil (bn)^a \rceil x} = \sum_{n=1}^{\infty} \left[z^{nx} - z^{(n+1)x}\right] \left\lfloor \frac{n^{\frac{1}{a}}}{b} \right\rfloor$$

Further, with a basic manipulations we arrive at equation (12). □

6. Generalised Reciprocal Fibonacci Series

6.1. Shah-Pingala Function

Definition 1. *For $q \in \mathbb{Z}\backslash\mathbb{Z}^-(\mathbb{N} \cup \{0\})$, we define Shah-Pingala function by the following equation*

$$S(q) = \sum_{n=1}^{\infty} \frac{n^q}{F_n}, \tag{13}$$

where F_n is n^{th} Fibonacci number (section 2.4).

The reciprocal Fibonacci constant is a special case of the new definition for $q = 0$ ($S(0) = 3.35988566243\ldots$).

Theorem 7. *Shah-Pingala function "$S(q)$" converges for all $q \in \mathbb{Z}\backslash\mathbb{Z}^-$.*

Proof. The n^{th} term is $\frac{n^q}{F_n}$. We know that

$$F_n > \left(\frac{2\varphi}{\sqrt{5}}\right)^n \Rightarrow (F_n)^{-1} < \left(\frac{2\varphi}{\sqrt{5}}\right)^{-n}.$$

We also know that $\lim\limits_{n\to\infty} \frac{n^q}{z^n} = 0$ for $q \in \mathbb{R} \wedge z > 1$. Hence,

$$0 \le \lim_{n\to\infty} \frac{n^q}{F_n} \le \lim_{n\to\infty} \frac{n^q}{\left(\frac{2\varphi}{\sqrt{5}}\right)^n} \Rightarrow \lim_{n\to\infty} \frac{n^q}{F_n} = 0\,.$$

Therefore, the necessary condition holds.
Furthermore, considering the ratio test for series convergence we have,

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(n+1)^q}{F_{n+1}}}{\frac{n^q}{F_n}}\right| = \lim_{n\to\infty} \left|\left(1 + \frac{1}{n}\right)^q \frac{F_n}{F_{n+1}}\right| = \frac{1}{\varphi} < 1$$

Since the limit $L < 1$, the series converges. □

6.2. Floor Reciprocal Fibonacci Function

Theorem 8. *Let $a, b \in \mathbb{R}^+$ and let F_n be Fibonacci sequence then,*

$$\sum_{n=1}^{\infty} \frac{1}{F_{\lfloor (bn)^a \rfloor}} = \sum_{n=1}^{\infty} \frac{\left\lfloor \left\lceil \frac{(n+1)^{\frac{1}{a}}}{b} \right\rceil - \left\lceil \frac{n^{\frac{1}{a}}}{b} \right\rceil \right\rfloor}{F_n}. \tag{14}$$

Proof. Substituting $m = 0$, and $k_n = \frac{1}{F_n}$ in equation (5), we get equation (14). □

6.3. Ceiling Reciprocal Fibonacci Function

Theorem 9. *Let $a, b \in \mathbb{R}^+$ and let F_n be Fibonacci sequence then,*

$$\sum_{n=1}^{\infty} \frac{1}{F_{\lceil (bn)^a \rceil}} = \sum_{n=1}^{\infty} \frac{\left\lfloor \left\lfloor \frac{n^{\frac{1}{a}}}{b} \right\rfloor - \left\lfloor \frac{(n-1)^{\frac{1}{a}}}{b} \right\rfloor \right\rfloor}{F_n}. \tag{15}$$

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Proof. Substituting $m = 0$, and $k_n = \frac{1}{F_n}$ in equation (7), we get equation (15). □

7. Corollaries

7.1. Corollaries of section 4

Corollary 1. *(F-Shah-Hurwitz Zeta function)*
For the "F-Shah-Hurwitz Zeta function" (a special case of F-Hurwitz zeta function at $a = \frac{1}{q}$, $q \in \mathbb{N}$, $b = 1$) and Hurwitz Zeta function $\zeta(s,t)$ the following equation holds true.

$$^{F}\zeta_1^{\frac{1}{q}}(s,t) = \sum_{n=0}^{\infty} \frac{1}{\left(\lfloor \sqrt[q]{n} \rfloor + t\right)^s} = \sum_{m=0}^{q-1} \sum_{k=0}^{m} \binom{q}{m} \binom{m}{k} (-t)^{m-k} \zeta(s-k,t) \tag{16}$$

or alternatively,

$$^{F}\zeta_1^{\frac{1}{q}}(s,t) = \int_0^\infty \frac{P(x,s,t,q)e^{-tx}}{1-e^{-x}} dx, \tag{17}$$

where,

$$P(x,s,t,q) = \sum_{m=0}^{q-1} \sum_{k=0}^{m} \binom{q}{m} \binom{m}{k} (-t)^{m-k} (\Gamma(s-k))^{-1} x^{s-k-1}.$$

Given the analytic continuation of the Hurwitz Zeta function, we observe that $^{F}\zeta_1^{\frac{1}{q}}(s,t)$ can be defined for $Re(s) < q$ and hence $^{F}\zeta_1^{\frac{1}{q}}(s)$ can be defined $\forall s \in \mathbb{C}$, such that $s \neq 1,2,\ldots,q$.

Proof. From section 4 we have

$$^{F}\zeta_b^{a}(s,t) = \sum_{n=0}^{\infty} \frac{1}{\left(\lfloor (bn)^a \rfloor + t\right)^s} = \sum_{n=0}^{\infty} \frac{\left[\left\lceil \frac{(n+1)^{\frac{1}{a}}}{b} \right\rceil - \left\lceil \frac{n^{\frac{1}{a}}}{b} \right\rceil\right]}{(n+t)^s}.$$

Substituting $a = \frac{1}{q}$ and $b = 1$ we get

$$^{F}\zeta_1^{\frac{1}{q}}(s,t) = \sum_{n=0}^{\infty} \frac{1}{\left(\lfloor \sqrt[q]{n} \rfloor + t\right)^s} = \sum_{n=0}^{\infty} \frac{(n+1)^q - n^q}{(n+t)^s}.$$

Considering the right hand side of the previous equation, we get,

$$\sum_{n=0}^{\infty} \frac{(n+1)^q - n^q}{(n+t)^s} = \sum_{n=0}^{\infty} \frac{\sum_{m=0}^{q-1} \binom{q}{m} n^m}{(n+t)^s} = \sum_{n=0}^{\infty} \sum_{m=0}^{q-1} \binom{q}{m} \frac{(n+t-t)^m}{(n+t)^s}$$
$$= \sum_{n=0}^{\infty} \sum_{m=0}^{q-1} \binom{q}{m} \frac{\sum_{k=0}^{m} \binom{m}{k} (n+t)^k (-t)^{m-k}}{(n+t)^s} = \sum_{m=0}^{q-1} \sum_{k=0}^{m} \binom{q}{m} \binom{m}{k} (-t)^{m-k} \sum_{n=0}^{\infty} \frac{1}{(n+t)^{s-k}}$$
$$= \sum_{m=0}^{q-1} \sum_{k=0}^{m} \binom{q}{m} \binom{m}{k} (-t)^{m-k} \zeta(s-k,t).$$

Finally, the alternate (integral representation) form of the equation can be obtained by substituting $\zeta(s-k,t) = \Gamma(s-k)^{-1} \int_0^\infty \frac{x^{s-k-1} e^{-tx}}{1-e^{x}} dx$ (from equation (1)). □

Corollary 2. *(C-Shah-Hurwitz Zeta function)*
For the "C-Shah-Hurwitz Zeta function" (a special case of C-Hurwitz zeta function at $a = \frac{1}{q}$, $q \in \mathbb{N}$, $b = 1$) and Hurwitz Zeta function $\zeta(s,t)$ the following equation holds true.

$$^{C}\zeta_1^{\frac{1}{q}}(s,t) = \sum_{n=0}^{\infty} \frac{1}{\left(\lceil \sqrt[q]{n} \rceil + t\right)^s} = \sum_{m=0}^{q-1} \sum_{k=0}^{m} (-1)^{q-m+1} \binom{q}{m} \binom{m}{k} (-t)^{m-k} \zeta(s-k,t) \tag{18}$$

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or alternatively

$$^{C}\zeta_1^{\frac{1}{q}}(s,t) = \int_0^\infty \frac{Q(x,s,t,q)e^{-tx}}{1-e^{-x}}dx. \tag{19}$$

where,

$$Q(x,s,t,q) = \sum_{m=0}^{q-1}\sum_{k=0}^{m}(-1)^{q-m+1}\binom{q}{m}\binom{m}{k}(-t)^{m-k}(\Gamma(s-k))^{-1}x^{s-k-1}$$

Given the analytic continuation of the Hurwitz Zeta function, we observe that $^{F}\zeta_1^{\frac{1}{q}}(s,t)$ *can be defined even for* $Re(s) < q$ *and hence* $^{F}\zeta_1^{\frac{1}{q}}(s)$ *can be defined* $\forall s \in \mathbb{C}$*, such that* $s \neq 1,2,\ldots,q$.

Proof. From section 4 we have

$$^{C}\zeta_b^a(s,t) = \sum_{n=0}^{\infty}\frac{1}{(\lceil (bn)^a\rceil + t)^s} = \sum_{n=0}^{\infty}\frac{\left[\left\lfloor\frac{n^{\frac{1}{a}}}{b}\right\rfloor - \left\lfloor\frac{(n-1)^{\frac{1}{a}}}{b}\right\rfloor\right]}{(n+t)^s}$$

Substituting $a = \frac{1}{q}$ and $b = 1$ we get

$$^{C}\zeta_1^{\frac{1}{q}}(s,t) = \sum_{n=0}^{\infty}\frac{1}{(\lceil\sqrt[q]{n}\rceil + t)^s} = \sum_{n=0}^{\infty}\frac{n^q-(n-1)^q}{(n+t)^s}$$

Consider the right hand side of the previous equation:

$$\sum_{n=0}^{\infty}\frac{n^q-(n-1)^q}{(n+t)^s} = \sum_{n=0}^{\infty}\frac{\sum_{m=0}^{q-1}(-1)^{q-m+1}\binom{q}{m}n^m}{(n+t)^s} = \sum_{n=0}^{\infty}\sum_{m=0}^{q-1}(-1)^{q-m+1}\binom{q}{m}\frac{(n+t-t)^m}{(n+t)^s}$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{q-1}(-1)^{q-m+1}\binom{q}{m}\frac{\sum_{k=0}^{m}\binom{m}{k}(n+t)^k(-t)^{m-k}}{(n+t)^s}$$
$$= \sum_{m=0}^{q-1}\sum_{k=0}^{m}(-1)^{q-m+1}\binom{q}{m}\binom{m}{k}(-t)^{m-k}\sum_{n=0}^{\infty}\frac{1}{(n+t)^{s-k}}$$
$$= \sum_{m=0}^{q-1}\sum_{k=0}^{m}(-1)^{q-m+1}\binom{q}{m}\binom{m}{k}(-t)^{m-k}\zeta(s-k,t).$$

Finally, the alternate (integral representation) form of the equation can be obtained by substituting $\zeta(s-k,t) = \Gamma(s-k)^{-1}\int_0^\infty \frac{x^{s-k-1}e^{-tx}}{1-e^x}dx$ (from equation (1)). □

Corollary 3. *(F-Shah-Riemann Zeta Function)*
For any a of the form $a = \frac{1}{q}, q \in \mathbb{N}$ *and* $b = 1$ *the following equation holds true.*

$$^{F}\zeta_1^{\frac{1}{q}}(s,1) = {^{F}\zeta_1^{\frac{1}{q}}}(s) = \sum_{n=1}^{\infty}\frac{1}{\lfloor\sqrt[q]{n}\rfloor^s} = \sum_{m=0}^{q-1}\binom{q}{m}\zeta(s-m) \tag{20}$$

or alternatively,

$$^{F}\zeta_1^{\frac{1}{q}}(s) = \int_0^\infty \frac{P(x,s,q)}{e^x-1}dx. \text{ where } P(x,s,q) = \sum_{t=0}^{q-1}(\Gamma(s-t))^{-1}\binom{q}{t}x^{s-t-1} \tag{21}$$

Furthermore, given the analytic continuation of the Riemann zeta function we observe that $^{F}\zeta_1^{\frac{1}{q}}(s)$ *can be defined for* $Re(s) < q$ *and hence* $^{F}\zeta_1^{\frac{1}{q}}(s)$ *can be defined* $\forall s \in \mathbb{C}$ *such that* $s \neq 1,2,\ldots,q$.

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Proof. Equation (20) can be obtained by substituting $t = 1$ in equation (16). Moreover, we may consider integral representation of the Riemann zeta function from equation (3) to obtain equation (21). □

Corollary 4. *(C-Shah-Riemann Zeta Function)*
For any a of the form $a = \frac{1}{q}, q \in \mathbb{N}$ and $b = 1$ the following equation holds true.

$$^{C}\zeta_1^{\frac{1}{q}}(s,1) = ^{C}\zeta_1^{\frac{1}{q}}(s) = \sum_{n=1}^{\infty} \frac{1}{\lceil \sqrt[q]{n} \rceil^s} = \sum_{m=0}^{q-1} (-1)^{q-m+1} \binom{q}{m} \zeta(s-m) \tag{22}$$

or alternatively,

$$^{C}\zeta_1^{\frac{1}{q}}(s) = \int_0^{\infty} \frac{Q(x,s,q)}{e^x - 1} dx; \text{ where } Q(x,s,q) = \sum_{t=0}^{q-1} \frac{(-1)^{q-t+1}}{\Gamma(s-t)} \binom{q}{t} x^{s-t-1} \tag{23}$$

Again, given the analytic continuation of the Riemann zeta function we observe that $^{C}\zeta_1^{\frac{1}{q}}(s)$ can be defined even for $Re(s) < q$ and hence $^{C}\zeta_1^{\frac{1}{q}}(s)$ can be define $\forall s \in \mathbb{C}$ such that $s \neq 1, 2, \ldots, q$.

Proof. Equation (22) can be obtained by substituting $t = 1$ in equation (18). Moreover, we may take integral representation of the Riemann zeta function from equation (3) to obtain equation (23). □

Remark 2. *All the series in corollaries 1-4 have poles at $s = 1, 2, \ldots, q$, but following can be simply observed for $q \neq 1$:*

$$^{F}\zeta_1^{\frac{1}{q}}(q,t) - ^{C}\zeta_1^{\frac{1}{q}}(q,t) = \sum_{m=0}^{q-2} \sum_{k=0}^{m} \binom{q}{m} \binom{m}{k} \zeta(q-k,t) \Big\{ (-t)^{m-k} + (-1)^{q-m} (-t)^{m-k} \Big\}. \tag{24}$$

and

$$^{F}\zeta_1^{\frac{1}{q}}(q) - ^{C}\zeta_1^{\frac{1}{q}}(q) = \sum_{n=1}^{\infty} \left(\frac{1}{\lfloor \sqrt[q]{n} \rfloor^q} - \frac{1}{\lceil \sqrt[q]{n} \rceil^q} \right) = \sum_{t=0}^{q-2} \binom{q}{t} \zeta(q-t) \Big[1 + (-1)^{q-t} \Big]. \tag{25}$$

This shows that even if the set of two series may individually have poles at $s = 1, 2, \ldots, q$, but their difference is convergent for $s = q$.

Remark 3. *The analogy used in Corollaries 1-4 can be implemented to different functions such as Lerch Zeta function.*

$$^{F}L_1^{\frac{1}{q}}(\lambda, s, t) = \sum_{m=0}^{q-1} \sum_{k=0}^{m} \binom{q}{m} \binom{m}{k} (-t)^{m-k} L(\lambda, s-k, t)$$

7.2. Corollaries of section 5

Corollary 5. *For $|z| < 1$, Polylogarithm $Li_s(z)$ and any $q \in \mathbb{N}$, the following equation holds true.*

$$\sum_{n=1}^{\infty} z^{\lfloor \sqrt[q]{n} \rfloor} = (z^{-1} - 1) Li_{-q}(z) - 1. \tag{26}$$

Proof. Equation (26) can be obtained by substituting $x = b = 1, a = \frac{1}{q}, q \in \mathbb{N}$ in equation (11). □

Corollary 6. *For $|z| < 1$, Polylogarithm $Li_s(z)$ and any $q \in \mathbb{N}$, the following equation holds true.*

$$\sum_{n=1}^{\infty} z^{\lceil \sqrt[q]{n} \rceil} = (1-z)Li_{-q}(z). \tag{27}$$

Proof. Equation (27) can be obtained by substituting $x = b = 1, a = \frac{1}{q}, q \in \mathbb{N}$ in equation (12). □

7.3. Corollaries of section 6

Corollary 7. *Let F_n be Fibonacci sequence, $q \in \mathbb{Z} \backslash \mathbb{Z}^-$ and let $S(q)$ denote Shah-Pingala function then the following equation holds true.*

$$\sum_{n=1}^{\infty} \frac{1}{F_{\lfloor \sqrt[q]{n} \rfloor}} = \sum_{n=1}^{\infty} \frac{n^q F_{n-2}}{\sum_{i=1}^{n-1} F_i^2} = \sum_{t=0}^{q-1} \binom{q}{t} S(t). \tag{28}$$

Proof. Equation (28) can be obtained by substituting $b = 1, a = \frac{1}{q}$, $q \in \mathbb{N}$ in equation (14). □

Corollary 8. *Let F_n be Fibonacci sequence, $q \in \mathbb{Z} \backslash \mathbb{Z}^-$ and let $S(q)$ denote Shah-Pingala function then the following equation holds true.*

$$\sum_{n=1}^{\infty} \frac{1}{F_{\lceil \sqrt[q]{n} \rceil}} = \sum_{n=1}^{\infty} \frac{n^q F_{n-1}}{\sum_{i=1}^{n} F_i^2} = \sum_{t=0}^{q-1} (-1)^{q-t+1} \binom{q}{t} S(t). \tag{29}$$

Proof. Equation (29) can be obtained by substituting $b = 1, a = \frac{1}{q}$, $q \in \mathbb{N}$ in equation (15). □

8. Plots and Zeros: F-Shah-Riemann Zeta and C-Shah-Riemann Zeta Functions

For this study, we consider all the available zeros of Riemann-zeta function i.e., the real zeros $-2n, n \in \mathbb{N}$ and the known complex zeros on the critical strip $\sigma + i \cdot t, 0 < \sigma < 1,\ t \in \mathbb{R}$, as the trivial solutions (already available zeros). Keeping that in view, we consider the negative solutions and the solutions whose imaginary parts are in the radius $(t-1, t+1)$ (imaginary part of solutions of the Riemann-zeta function) as trivial for the new zeta functions. Moreover, positive real zeros and the solutions whose imaginary parts are not in the radius $(t-1, t+1)$ are considered as non-trivial zeros. In all six of the plots we have focused on the behaviour of the functions near their non-trivial zeros.

8.1. Zeros of F-Shah-Riemann Zeta Function

In this subsection, utilizing the right hand side of the equation (20) we achieve the solutions (given in table 1) of F-Shah-Riemann zeta function. We have further given three figures 1, 2 and 3, and they are plotted in the intervals of the non-trivial (positive real) zeros and the poles of F-Shah-Riemann zeta function for $q = 2$, $q = 3$ and $q = 4$ respectively.

The function has q poles, $q-1$ positive real zeros (non-trivial zeros) and infinite negative real zeros (each zero corresponding to each negative even integer).
The complex zeros are achieved by implementing the Newton-Raphson method at point $x_0 = \frac{1}{2} + i \cdot t$ for different available values of t.

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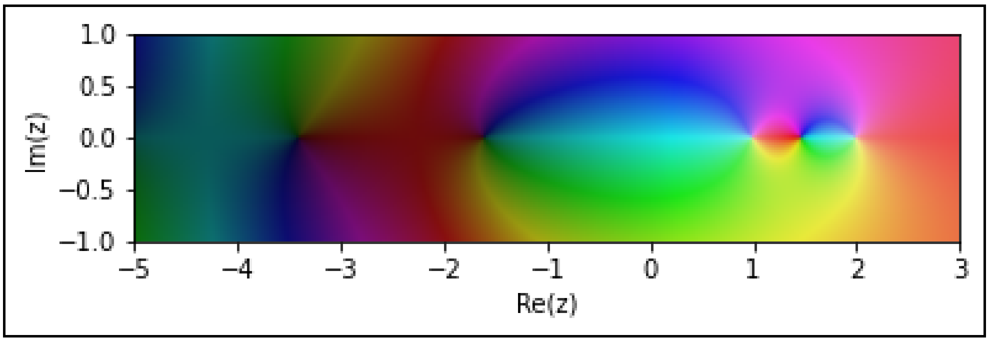

Figure 1. F-Shah-Riemann zeta function for $q = 2$: ${}^{F}\zeta_1^{\frac{1}{2}}(s)$. Two poles of the function are observed as $s = 1$ and $s = 2$, whereas a positive real root can be observed at $s \approx 1.473414717168$.

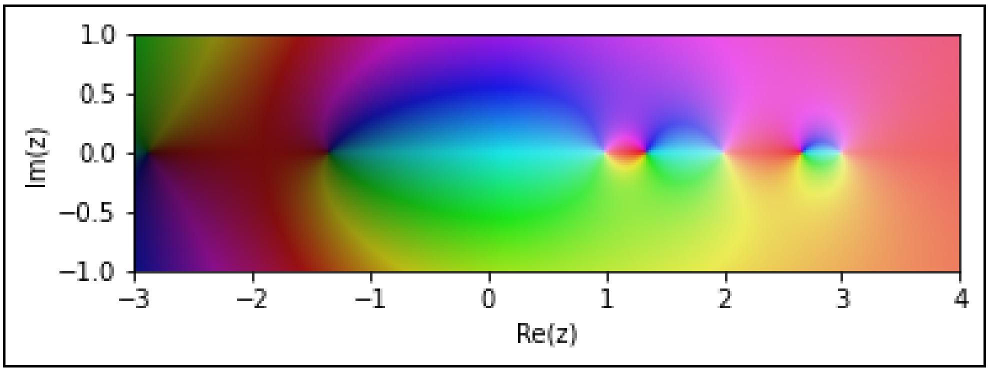

Figure 2. F-Shah-Riemann zeta function for $q = 3$: ${}^{F}\zeta_1^{\frac{1}{3}}(s)$. Three poles of the function are observed as $s = 1, s = 2$ and $s = 3$, whereas two positive real roots can be observed at $s \approx 1.34672733238$ and $s = 2.675764968478$.

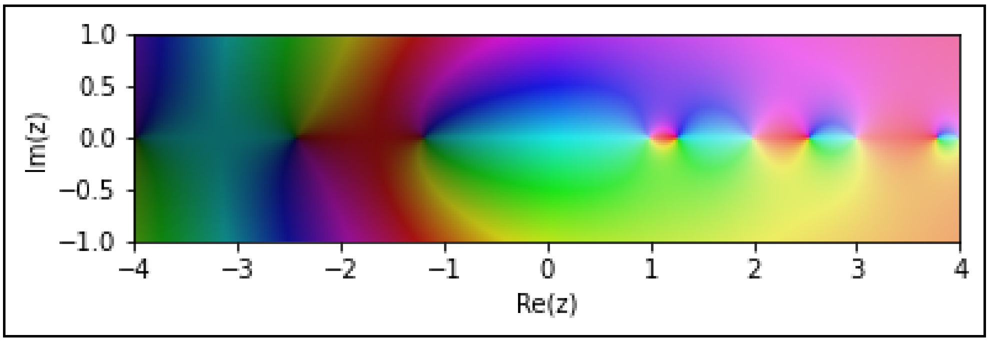

Figure 3. F-Shah-Riemann zeta function for $q = 4$: ${}^{F}\zeta_1^{\frac{1}{4}}(s)$. Four poles of the function are observed as $s = 1, s = 2, s = 3$ and $s = 4$, whereas three positive real roots can be observed at $s \approx 1.271435903075, s \approx 2.551800083744$ and $s \approx 3.796568180266$.

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Table 1. All the positive (non-trivial) real zeros and some of the complex and negative (trivial) real zeros of F-Shah-Riemann zeta function for values of $q = 2, q = 3$ and $q = 4$ up to 12 decimal places.

Values of q	Complex Zeros	Non-Trivial Real Zeros	Trivial Real Zeros
q = 2	1.247595281027 + 14.148570425918i	1.473414717168	-1.606882014624
	1.279113135722 + 21.012442575688i	-	-3.4037619981310
q = 3	1.964049664859 + 14.165353520342i	1.346727332380	-1.346011820212
	2.032696553488 + 21.001910485581i	2.675764968478	-2.878069065724
	2.062342325067 + 25.053176875792i	-	-4.623564926958
q = 4	2.653604262294 + 14.184700388061i	1.271435903075	-1.192688707675
	2.763852133144 + 20.991263644295i	2.551800083744	-2.423972912304
	2.811381226897 + 25.077915228667i	3.796568180266	-3.975616047809
	2.852031019544+ 30.316385253077i	-	-5.736688777159

8.2. Zeros of C-Shah-Riemann Zeta Function

In this subsection, utilizing the right hand side of the equation (22) we achieve the solutions (given in table 2) of C-Shah-Riemann zeta function. We have further given three figures 4, 5 and 6, and they are plotted in the intervals of the non-trivial (positive real) zeros and the poles of F-Shah-Riemann zeta function for $q = 2$, $q = 3$ and $q = 4$ respectively.

The function has q poles, $(q-1)$ pairs of non-trivial of complex zeros ($2(q-1)$ zeros), which were observed in the plots. There are infinite real zeros (each zero corresponds to each negative even integer) with no non-trivial (positive) real zero.
The trivial complex zeros are achieved by implementing the Newton-Raphson method with initial guess $x_0 = \frac{1}{2} + i \cdot t$ for different available values of t, whereas, the non trivial complex zeros are obtained by considering an initial guess x_0 in the converging region of the plot (for example, we considered $x_0 = 0.8 + 2i$ for C-Shah-Riemann zeta function for $q = 2$).

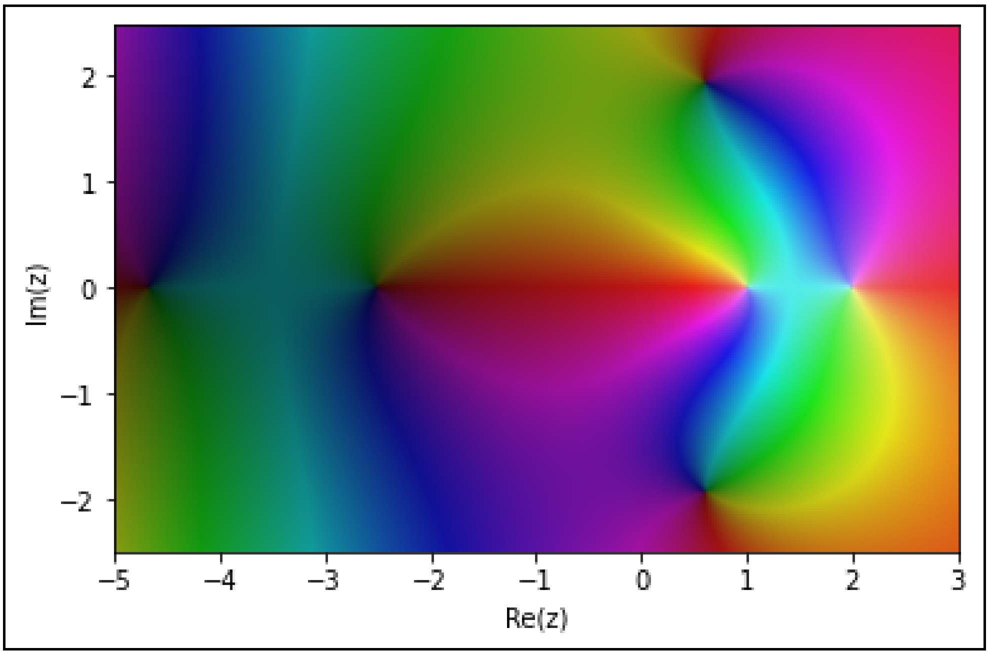

Figure 4. C-Shah-Riemann zeta function for $q = 2$: ${}^{C}\zeta_{1}^{\frac{1}{2}}(s)$. Two poles of the function are observed as $s = 1$ and $s = 2$, whereas a pair of non-trivial complex root can be observed at $s \approx 0.6078048160 \pm 1.9350010902i$.

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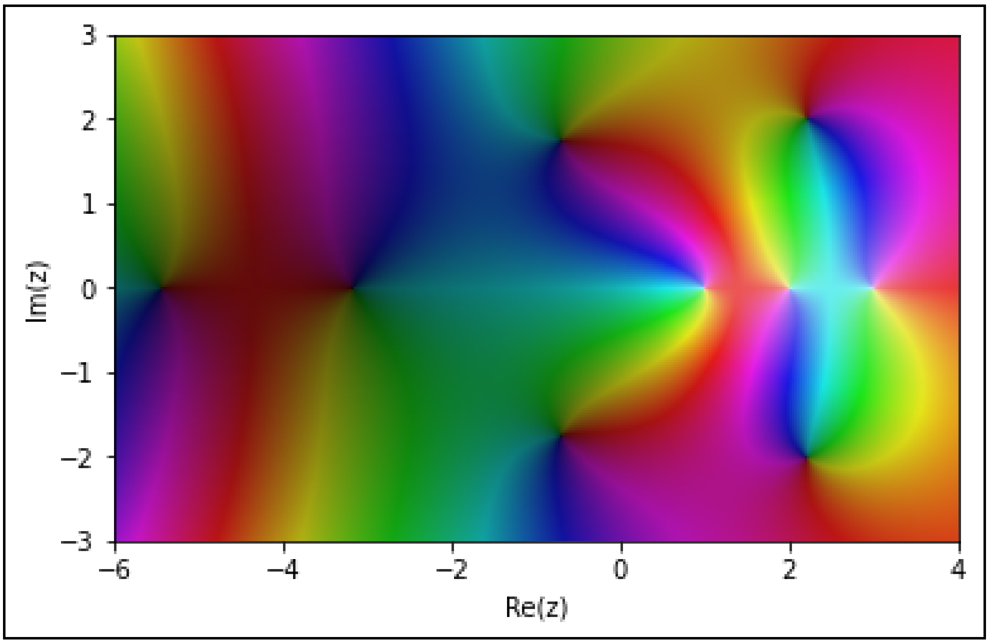

Figure 5. C-Shah-Riemann zeta function for $q = 3$: ${}^{C}\zeta_1^{\frac{1}{3}}(s)$. Three poles of the function are observed as $s = 1, s = 2$ and $s = 3$, whereas two pairs of non-trivial complex root can be observed at $s \approx -0.6997539191 \pm 1.7534110356i$ and $s \approx 2.2160159281 \pm 2.0194666275i$.

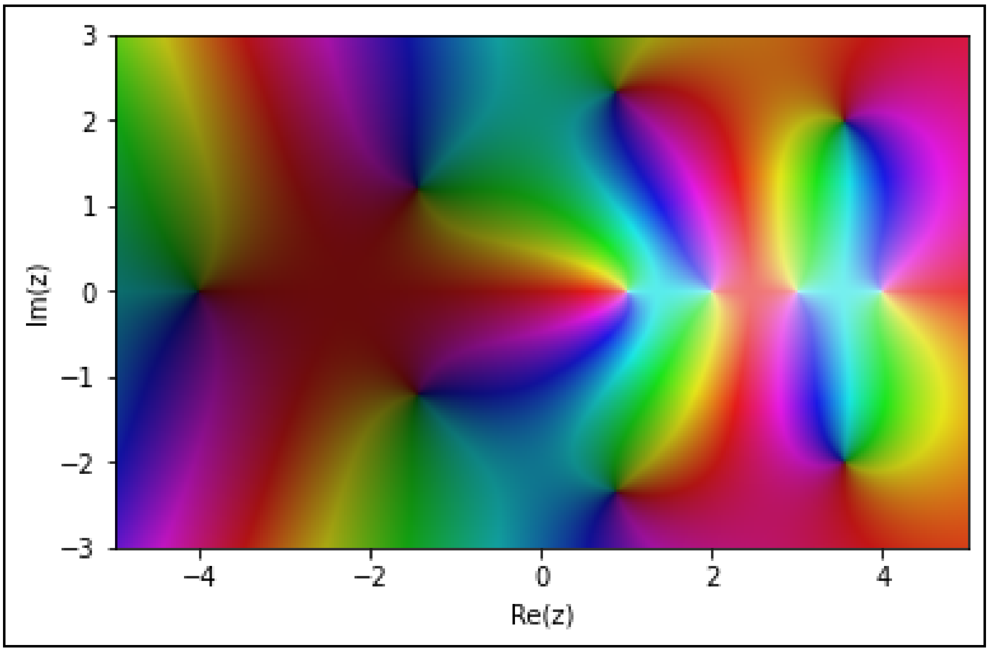

Figure 6. C-Shah-Riemann zeta function for $q = 4$: ${}^{C}\zeta_1^{\frac{1}{4}}(s)$. Four poles of the function are observed as $s = 1, s = 2, s = 3$ and $s = 4$, whereas three pairs of non-trivial complex root can be observed at $s \approx -1.4515176098 \pm 1.2122471911i$, $s \approx 0.8794024170 \pm 2.3478232248i$ and $s \approx 3.5554537875 \pm 1.9983004535i$.

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Table 2. Some of the non-trivial and trivial complex and real zeros of ${}^{C}\zeta_1^{\frac{1}{q}}(s)$ for values of $q = 2, q = 3$ and $q = 4$ up to 10 decimal places.

Values of q	Trivial Complex Zeros	Non-Trivial Complex Zeros	Real Zeros
q = 2	2.0460386041 + 14.0894409779i	0.6078048160 + 1.9350010902i	-2.514994862054
	1.9443544646 + 21.0650038610i	0.6078048160 - 1.9350010902i	-4.675064985091
q = 3	3.2683247579 + 14.0859069231i	-0.6997539191 + 1.7534110356i	-3.180342471496
	3.1531129164 + 21.0656065389i	2.2160159281 + 2.0194666275i	-5.450484894914
	3.1027440708 + 24.9358435149i	2.2160159281 - 2.0194666275i	-7.577495596468
q = 4	4.3760513879 + 14.0924715621i	-1.4515176098 + 1.2122471911i	-4.035452332431
	4.2716503066 + 21.0555557135i	0.8794024170 + 2.3478232248i	-6.316237379469
	4.2263218286 + 24.9427024675i	3.5554537875 + 1.9983004535i	-8.456827453077
	4.1822154523 + 30.5605816189i	3.5554537875 - 1.9983004535i	-10.54593349392

9. Results for Specific Values

9.1. Specific values - section 4

For $a = b = 1$, $'F'$ and $'C'$ generalised series reduce to the original series. (Consider F-Hurwitz zeta function and C-Riemann zeta for example)

$$ {}^{F}\zeta_1^1(s,t) = \sum_{n=0}^{\infty} \frac{1}{\left\lfloor (1\cdot n)^1 + t \right\rfloor^s} = \sum_{n=0}^{\infty} \frac{1}{(n+t)^s} = \zeta(s,t) $$

$$ {}^{C}\zeta_1^1(s) = \sum_{n=0}^{\infty} \frac{1}{\left(\left\lceil (1\cdot n)^1 \right\rceil + 1\right)^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) $$

9.2. Specific values - section 5

Substituting $a = b = x = 1$ in equation (11) we get

$$ \sum_{n=1}^{\infty} z^n = Li_0(z) = (z^{-1} - 1)Li_1(z) - 1 $$

or substituting $a = b = x = 1$ in equation (12) we obtain

$$ \sum_{n=1}^{\infty} z^n = Li_0(z) = (1 - z)Li_1(z) $$

both of which hold true.

9.3. Specific values - section 6

For $a = b = 1$ in equations (14) and (15) we get $S(0) \approx 3.35988566243\ldots$.

10. Miscellaneous Results

Let the two functions be defined as ${}^{F}g_b^a(n) = \left\lceil \frac{(n+1)^{\frac{1}{a}}}{b} \right\rceil - \left\lceil \frac{n^{\frac{1}{a}}}{b} \right\rceil$ and ${}^{C}g_b^a(n) = \left\lfloor \frac{n^{\frac{1}{a}}}{b} \right\rfloor - \left\lfloor \frac{(n-1)^{\frac{1}{a}}}{b} \right\rfloor$ which yield the number of repetition of $\lfloor (bn)^a \rfloor$ and $\lceil (bn)^a \rceil$ respectively. Then for particular values of a and b we provide the following assumptions (given in tabel 3) which are made solely by intuitions and have been checked and verified to be true for different values (of n) using the Python programming language.

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Table 3. Some observed equivalent results for ${}^{F}g_b^a(n)$ and ${}^{C}g_b^a(n)$

$(\frac{1}{a}, b)$	Observed equivalent result for ${}^{F}g_b^a(n)$	Observed equivalent result for ${}^{C}g_b^a(n)$
$(2,2)$	$\left\lceil \frac{(n+1)^2}{2} \right\rceil - \left\lceil \frac{n^2}{2} \right\rceil = 2\left\lfloor \frac{n}{2} \right\rfloor + 1$	$\left\lfloor \frac{n^2}{2} \right\rfloor - \left\lfloor \frac{(n-1)^2}{2} \right\rfloor = 2\left\lfloor \frac{n}{2} \right\rfloor$
$(2,3)$	$\left\lceil \frac{(n+1)^2}{3} \right\rceil - \left\lceil \frac{n^2}{3} \right\rceil = 2\left\lfloor \frac{n}{3} \right\rfloor + 1$	$\left\lfloor \frac{n^2}{3} \right\rfloor - \left\lfloor \frac{(n-1)^2}{3} \right\rfloor = \left\lfloor \frac{2n}{2} \right\rfloor$
$(2,4)$	$\left\lceil \frac{(n+1)^2}{4} \right\rceil - \left\lceil \frac{n^2}{4} \right\rceil = \left\lceil \frac{n}{2} \right\rceil + (-1)^n$	$\left\lfloor \frac{n^2}{4} \right\rfloor - \left\lfloor \frac{(n-1)^2}{4} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor$
$(3,2)$	$\left\lceil \frac{(n+1)^3}{2} \right\rceil - \left\lceil \frac{n^3}{2} \right\rceil = \frac{3n(n+1)}{2} + \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n-1}{2} \right\rfloor$	$\left\lfloor \frac{n^3}{2} \right\rfloor - \left\lfloor \frac{(n-1)^3}{2} \right\rfloor = \frac{3n(n-1)}{2} + \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n-1}{2} \right\rfloor$
$(3,3)$	$\left\lceil \frac{(n+1)^3}{3} \right\rceil - \left\lceil \frac{n^3}{3} \right\rceil = n(n+1) + \left\lfloor \frac{n}{3} \right\rfloor - \left\lfloor \frac{n-1}{3} \right\rfloor$	$\left\lfloor \frac{n^3}{2} \right\rfloor - \left\lfloor \frac{(n-1)^3}{2} \right\rfloor = n(n-1) + \left\lfloor \frac{n}{3} \right\rfloor - \left\lfloor \frac{n-1}{3} \right\rfloor$
$(3,4)$	$\left\lceil \frac{(n+1)^3}{4} \right\rceil - \left\lceil \frac{n^3}{4} \right\rceil = \left\lfloor \frac{3n(n+1)}{4} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n-1}{2} \right\rfloor$	$\left\lfloor \frac{n^3}{4} \right\rfloor - \left\lfloor \frac{(n-1)^3}{4} \right\rfloor = \left\lfloor \frac{3n(n-1)}{4} \right\rfloor + \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n-1}{2} \right\rfloor$

One can go further with $\frac{1}{a} = 4, 5, 6, \ldots$ and different values of b for both functions. In general, our assumptions for $\left(\frac{1}{q}, 2\right)$ are:

(I) For number of repetition of $\left\lfloor \sqrt[q]{2n} \right\rfloor$

$$ {}^{F}g_2^{\frac{1}{q}}(n) = \left\lceil \frac{(n+1)^q}{2} \right\rceil - \left\lceil \frac{n^q}{2} \right\rceil = \frac{q(q-1)}{2} \left\{ \sum_{i=1}^{n} \sum_{t=(\lceil \frac{q}{2} \rceil - \lfloor \frac{q}{2} \rfloor)}^{q-2} \left(\frac{i^t \left(1 + (-1)^{q-t}\right)}{2} \right) \right\} + y_n \tag{30} $$

where,

$$ y_n = \begin{cases} 2\left\lfloor \frac{n}{2} \right\rfloor + 1, & q \; even \\ \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n-1}{2} \right\rfloor, & q \; odd \end{cases} $$

(II) For number of repetition of $\left\lceil \sqrt[q]{2n} \right\rceil$

$$ {}^{C}g_2^{\frac{1}{q}}(n) = \left\lfloor \frac{n^q}{2} \right\rfloor - \left\lfloor \frac{(n-1)^q}{2} \right\rfloor = \frac{q(q-1)}{2} \left\{ \sum_{t=\lceil \frac{q}{2} \rceil - \lfloor \frac{q}{2} \rfloor}^{q-2} \left(\sum_{i=1}^{n-1} \left(\frac{i^t \left(1 + (-1)^{q-t}\right)}{2} \right) \right) \right\} + z_n \tag{31} $$

where,

$$ z_n = \begin{cases} 2\left\lfloor \frac{n}{2} \right\rfloor, & q \; even \\ \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n-1}{2} \right\rfloor, & q \; odd \end{cases} $$

These assumptions have been verified for the values of $q = 1, 2, 3, 4, 5$ using the python programming language.

Remark 4. *If both of these assumptions hold true then the following result holds:*

$$ \zeta(s) = {}^{F}\zeta_2^{\frac{1}{q}}(s) - {}^{C}\zeta_2^{\frac{1}{q}}(s). \tag{32} $$

11. Open Problems

In this section we propose the following open problems for future work.

11.1. Problem 1

Do the assumptions (30) and (31) hold true? Can the equality be proven for general case? If these assumptions hold true, one can go on to find a general formula for the equivalents of ${}^{F}g_k^{\frac{1}{q}}(s)$ and ${}^{C}g_k^{\frac{1}{q}}(s)$.

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11.2. Problem 2

For both $'C'$ and $'F'$ generalised series, the respective corresponding series on the right-hand side (the equivalent series) are observed to converge faster to the particular values, can it be proven mathematically?

11.3. Problem 3

Is it possible to obtain Euler-product formulae for "F-Shah-Riemann zeta function" and "C-Shah-Riemann zeta function"?

11.4. Problem 4

Can the "Floor-Ceiling" and "Ceiling-Floor" theorems be implemented on integrals?

11.5. Problem 5

Considering the vast number of available infinite series, studying, analysing and providing results for all of them is not possible in the scope of single article. Hence, we discussed just few of the results which could be derived from the discussed theorems and corollaries.
Therefore, we put forth the final open problem of this series of two papers for future studies to implement our results to different available infinite series (i.e., series involving (1) exponential function, (2) logarithmic function, (3) trigonometric functions, (4) different Dirichlet functions (Lerch zeta function, Dirichlet eta function), (5) Taylor Series or any results involving infinite series).
To inspire future studies, we list a few examples for reference:
(1) For the the exponential function e^z:

$$\sum_{n=0}^{\infty} \frac{z^{\lfloor \sqrt{n} \rfloor}}{\lfloor \sqrt{n} \rfloor !} = (2z+1)e^z$$

or (2) for the Polylogarithm $Li_s(z)$:

$$\sum_{n=1}^{\infty} \frac{z^{\lfloor \sqrt{n} \rfloor}}{\lfloor \sqrt{n} \rfloor^s} = 2Li_{s-1}(z) + Li_s(z)$$

or (3) for Hyperbolic Functions:

$$\sum_{n=1}^{\infty} \frac{z^{(2\lfloor \sqrt{n} \rfloor + 1)}}{(2\lfloor \sqrt{n} \rfloor + 1)!} = z\cosh(z)$$

or (4) for any function $f(x)$ (using the Maclaurin series) :

$$\sum_{n=0}^{\infty} \frac{f^{\lfloor \sqrt{n} \rfloor}(0) x^{\lfloor \sqrt{n} \rfloor}}{(\lfloor \sqrt{n} \rfloor)!} = 2xf'(x) + f(x)$$

12. Concluding Remarks

In this paper, we prove the "Floor-Ceiling" and "Ceiling-Floor" theorems (of part I [13]) for infinite series and take them as a base to provide new results involving zeta functions and Fibonacci numbers (in terms of theorems and corollaries). Further we provide some zeros of the newly derived zeta functions and plot them in complex plane using the concept of domain colouring.

The "Floor-Ceiling" and "Ceiling-Floor" theorems can potentially develop an entire field in Mathematics where, using them, one can derive hundreds, if not thousands, hitherto unknown results (such as Shah-Pingala formula in part I [13] or Shah-Hurwitz and Shah-Riemann zeta functions in part II) involving finite and infinite series. With the availability of those new results, one can further analyse their patterns and behaviour in domains of real and complex analyses and find their applications in different advanced fields as shown earlier [2–5].

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